**2.4 Probability of an Event**

Definition 10 (Probability)

The probability of event A is the sum of the weights of all the outcomes in A (which range from 0 to 1). Therefore,

If is a sequence of mutually exclusive events, then

There are three common ways of assigning probabilities to the outcomes of an experiment:

1. Empirically
2. Classical Approach
3. Subjectively

**Empirical Probability**

To calculate empirical probability of event A, repeat the experiment a large number of times and count the number of times event A actually occurs. Based on these observed results, P(A) is approximated as the relative frequency of event A.

Example 20

There is a coin with two sides, heads and tails. It is unknown whether the coin is fair or not. The coin is tossed 1000 times, and 405 times heads occurred. What can you estimate is the probability of obtaining heads when the coin is tossed?

**Classical Probability**

The classical approach to probability assumes that all outcomes in the sample space are equally likely. This assumption can be met if you roll a fair die, flip a fair coin, or randomly select 1 person out of a class of 20 students, for example.

Assuming that each outcome is equally likely, the probability of event A can be defined as the number of ways the event A can occur, divided by the number of outcomes in the sample space.

Example 21

A fair coin is tossed two times. Find the probability that exactly one head occurs.

**Subjective Probability**

Subjective probability assigns the probability of an outcomes based on one’s life experience. You use your knowledge of the relevant circumstances to assign a probability to a specific event.

Example 22

Experience with a particular stock suggests that there is a 90% chance that the share price will increase tomorrow.

Example 23

Two individuals are selected at random and without replacement from a population of three men and two women. Find the following probabilities using the classical approach to probability.

1. P(sample contains one man)
2. P(sample contains two men)
3. P(sample contains at least one man)
4. P(sample contains at most one man)

**2.5 Additive Rule**

Theorem 5

If A and B are two events, then

A diagram of a diagram of a diagram

Description automatically generated with medium confidence

Corollary 1

1. If A and B are mutually exclusive events, then
2. Given two events A and B, then

Example 24

Given that the probability of event A is 0.7, the probability of event B is 0.4, and the probability of both events A and B occurring together is 0.3.

1. Find the probability of either A or B occurring.
2. Find the probability that event A occurs, and event B does not occur.

Theorem 6

For three events A, B, and C,

A diagram of a green and yellow circle

Description automatically generated

Theorem 7

If A and are complementary, then

A green circle with a letter a

Description automatically generated

Example 25

Given , determine:

Example 26

One item is selected at random from the following population:

|  |  |
| --- | --- |
| Defects | Number of Items |
| None | 750 |
| Only type A | 20 |
| Only type B | 50 |
| Only type C | 70 |
| Only types A and B | 30 |
| Only types A and C | 40 |
| Only types B and C | 30 |
| Types A and B and C | 10 |
| Total | 1000 |
|  |  |

Determine the following:

1. What is the complement of ?

**2.6 Conditional Probability, Independence, and the Product Rule**

Definition 11 (Conditional Probability)

The **conditional probability** of B, given A, denoted by , is defined by:

Example 27

One individual is selected at random from the following population of 900 people.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Employed | Unemployed | Totals |
| Male | 460 | 40 | 500 |
| Female | 140 | 260 | 400 |
| Totals | 600 | 300 | 900 |

Determine .

Example 28

There is an 83% chance that a flight will depart on time, an 82% chance that a flight will arrive on time, and a 78% chance that a flight will both depart and arrive on time.

1. Find the probability that a flight arrives on time, given that it departed on time.
2. Find the probability that a flight departed on time, given that it arrived on time.

Theorem 8 (Law of Multiplication)

For anu events A, B, that can both occur, , where .

Note this is a consequence of the definition of .

Theorem 9 (Law of Multiplication for k Events)

The law of multiplication can be extended for a series of k events:

Example 29

Two balls are drawn randomly, one by one and without replacement, from a population containing 5 red and 4 white balls.

1. List all possible outcomes of the experiment where the balls are distinguished only by color.
2. Use the law of multiplication to assign a probability to each outcome.
3. Find the probability that the sample contains exactly one red ball.

Definition 12 (Independent Events)

Two events A and B are **independent** if and only if

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

Theorem 10 (Independent Events)

Two events A and B are **independent** if and only if .

Example 30

A small town has one fire engine and one ambulance. Let A be the event that the fire engine is available when needed and let B be the event that the ambulance is available when needed. Assume A and B are independent with P(A) = 0.96 and P(B) = 0.92. Determine:

Example 31

One individual is selected at random from the following population.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | Smoking |  |  |
| Hypertension | None | Moderate | Heavy | Total |
| Present | 300 | 180 | 120 | 600 |
| Absent | 200 | 120 | 80 | 400 |
| Total | 500 | 300 | 200 | 1000 |

Let A be the event that the selected individual is a moderate smoker and B be the event that the selected individual has hypertension. Determine whether A and B are independent events.

Example 32

When sampling without replacement from a large population, the successive draws are approximately independent.

Consider a population of 500 red balls and 400 white balls. Let be that the first ball selected is white and be that the second ball selected is red.

1. If two balls are selected without replacement from this population, what is ?
2. If two balls are selected with replacement from this population, what is ?

Theorem 11

If A and B are independent events, then each of the following pairs of events are also independent:

Proof that are independent:

Recall the law of multiplication for k events:

Theorem 12

If are independent events. Then

**2.7 Bayes’ Rule**

We can write an event A as the union of two mutually exclusive events:

Therefore, the probability of event A can be calculated as

Applying the law of multiplication to each of the intersections, we can write:

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Description automatically generated

Theorem 13 (Total Probability or Rule of Elimination)

If the events constitute a partition of the sample space S such that for , then for any event A of S,

A diagram of a circle with a blue circle and a blue circle with white lines

Description automatically generated

Example 33

In a certain assembly plant, three machines, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now suppose that a finished product is randomly selected. What is the probability that it is defective?

Suppose instead of asking for P(A), we need to calculate . This can be answered using Bayes’ theorem.

Theorem 14 (Bayes’ Rule)

If the events constitute a partition of the sample space S such that for , then for any event A in S such that .

Example 34

A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. Plans 1, 2, and 3 are used for 30%, 20% and 50% of the products respectively. The defective rate is different for the three procedures as follows:

* There is a 0.01 chance the product is defective using plan 1
* There is a 0.03 chance the product is defective using plan 2
* There is a 0.02 chance the product is defective using plan 3

If a random product was selected and found to be defective, which plan was most likely used and is, thus, responsible?